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SINGULAR SOLUTIONS IN AN AXISYMMETRIC FLOW OF A MEDIUM OBEYING THE DOUBLE SHEAR MODEL

S. E. Alexandrov

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An asymptotic analysis of equations of an axisymmetric flow of a rigid-plastic material obeying the double shear model in the vicinity of surfaces with the maximum friction is performed. It is shown that the solution is singular if the friction surface coincides with the envelope of the family of characteristics. A possible character of the behavior of singular solutions in the vicinity of surfaces with the maximum friction is determined. In particular, the equivalent strain rate in the vicinity of the friction surface tends to infinity in an inverse proportion to the square root from the distance to this surface. Such a behavior of the equivalent strain rate is also observed in the classical theory of plasticity of materials whose yield condition is independent of the mean stress.

Key words: *singularity, friction, double shear model, plasticity.*

The double shear model [1] belongs to models of the flow of rigid-plastic materials whose yield condition depends on the mean stress. Apparently, such a model including kinematics was first proposed in [2]. A brief review of models of this kind can be found in [3]. These models are generalizations of the classical theory of an ideal rigid-plastic material and are used to describe the motion of loose materials [1, 2] and deformation of some metallic alloys [4, 5].

In the classical theory of plasticity, singular solutions can arise in the vicinity of the maximum friction surfaces and surfaces with velocity discontinuities [6–8]. The maximum friction surface, which is a contact surface between a rigid tool and a deformable material, is determined by equality of specific friction forces due to slipping and the yield point due to pure shear. If the system of equations is hyperbolic, the maximum friction surface coincides with the characteristic or envelope of the family of characteristics; singularity in the solution appears if the friction surface coincides with the envelope. The law of the maximum friction in this formulation is used in the theory of plasticity with the yield condition depending on the mean stress [9–12]. Solutions of particular problems in the plane-strain state show that the asymptotic behavior of solutions in the vicinity of the maximum friction surface within the framework of such plasticity theories depends on the material model [11]. A typical feature of plane flows of materials obeying the model developed in [1] is the possibility of origination of singular solutions [12]; their asymptotic behavior in the vicinity of the maximum friction surfaces coincides with the corresponding solutions obtained within the framework of the classical plasticity [7, 8]. Thus, it seems of interest to find the possibility of origination of singular solutions in axisymmetric flows of materials obeying the model developed in [1] and, if such solutions exist, to find their asymptotic behavior in the vicinity of the friction surface.

The static equations of the model developed in [1] consist of equilibrium equations and the Mohr–Coulomb yield condition. In studying axisymmetric flows, one can naturally use cylindrical coordinates r, θ, z, in which the projection of velocity is $u_\theta = 0$, the stress $\sigma_{\theta\theta}$ is one of the principal stresses, and all derivatives with respect to θ equal zero. Without loss of generality, we can assume that $\sigma_3 = \sigma_{\theta\theta}$ and $\sigma_1 \geqslant \sigma_2$, where σ_1, σ_2, and σ_3 are the principal stresses. As the yield condition is singular, there exist several flow regimes. The cross section of the yield surface by the plane $\sigma = \mathrm{const}$ [$\sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3$ is the mean stress] is shown in Fig. 1. The most interesting

Institute of Problems in Mechanics, Russian Academy of Sciences, Moscow 119526; sergeyaleksandrov@yahoo.com. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 5, pp. 180–186, September–October, 2005. Original article submitted December 22, 2004.

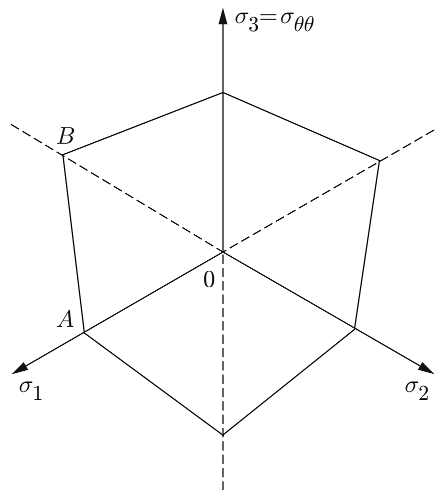

Fig. 1

axisymmetric flows encountered in most problems correspond to regimes A and B (Fig. 1). These regimes will be considered in the present paper. The yield condition for these regimes can be written as

$$\sigma_1(1+\sin\varphi) = 2c\cos\varphi + \sigma_2(1-\sin\varphi), \qquad \sigma_2 = \sigma_{\theta\theta} \tag{1}$$

for regime A and

$$\sigma_1(1+\sin\varphi) = 2c\cos\varphi + \sigma_2(1-\sin\varphi), \qquad \sigma_1 = \sigma_{\theta\theta} \tag{2}$$

for regime B. Here φ is the angle of internal friction and c is the adhesion coefficient. In terms of the stress-tensor components in the cylindrical coordinates σ_{rr}, σ_{zz}, $\sigma_{\theta\theta}$, and σ_{rz}, Eqs. (1) and (2) are written in the form

$$\begin{gathered}(\sigma_{rr}+\sigma_{zz})\sin\varphi + [(\sigma_{rr}-\sigma_{zz})^2 + 4\sigma_{rz}^2]^{1/2} = 2c\cos\varphi,\\ 2\sigma_{\theta\theta} = \sigma_{rr}+\sigma_{zz} - \varepsilon[(\sigma_{rr}-\sigma_{zz})^2 + 4\sigma_{rz}^2]^{1/2},\end{gathered} \tag{3}$$

where $\varepsilon = 1$ for regime A and $\varepsilon = -1$ for regime B. We introduce an angle ψ between the r axis and the first principal stress, which is counted from the r axis counterclockwise. Then, the yield condition (3) can be satisfied by assuming that [1]

$$\begin{gathered}\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{zz} = -p - q\cos 2\psi, \quad \sigma_{rz} = q\sin 2\psi, \quad \sigma_{\theta\theta} = -p - \varepsilon q,\\ p = -(\sigma_{rr}+\sigma_{zz})/2, \qquad q = p\sin\varphi + c\cos\varphi.\end{gathered} \tag{4}$$

The equilibrium equations have the form

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0, \qquad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \tag{5}$$

Substituting Eqs. (4) into Eqs. (5), we can obtain a system of equations with respect to p and ψ. This system is hyperbolic, and the angles of inclination of characteristics to the r axis are determined by the formulas (see, e.g., [1])

$$\phi = \psi \pm (\varphi/2 + \pi/4). \tag{6}$$

Let α be an angle between the r axis and the tangential line to the friction surface ω at a certain point M of this surface (Fig. 2). Then, it follow from the definition of the maximum friction law that $\phi = \alpha$ during slipping. From Eq. (6), we obtain the following equation on the friction surface:

$$\psi = \alpha - \varphi/2 - \pi/4. \tag{7}$$

For certainty, we choose the upper sign in Eq. (6). The second case can be examined in a similar manner. We introduce a local orthogonal coordinate system x_1, x_3 at the point M so that the x_1 axis is directed tangentially to

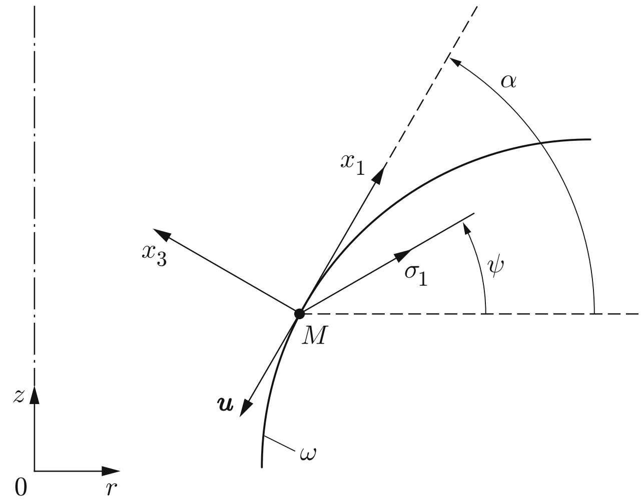

Fig. 2

the friction surface and the x_3 axis is directed inward the deformable material (Fig. 2). Without loss of generality, the tool can be assumed to be motionless. Then, the velocity vector $\boldsymbol{u}$ at the point M is directed along the x_1 axis. It follows from Eq. (7) that the angle $\alpha - \psi$ depends only on material properties and not on the geometry of a particular problem.

We substitute Eqs. (4) into Eqs. (5) and pass to differentiation with respect to x_1 and x_3 by the formulas

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x_1} \cos\alpha - \frac{\partial}{\partial x_3} \sin\alpha, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial x_1} \sin\alpha + \frac{\partial}{\partial x_3} \cos\alpha. \tag{8}$$

As a result, we obtain

$$\begin{gathered}
[\cos(\alpha - 2\psi)\sin\varphi - \cos\alpha]\,\frac{\partial p}{\partial x_1} + [\sin\alpha - \sin(\alpha - 2\psi)\sin\varphi]\,\frac{\partial p}{\partial x_3} \\
+ 2q\sin(\alpha - 2\psi)\,\frac{\partial \psi}{\partial x_1} + 2q\cos(\alpha - 2\psi)\,\frac{\partial \psi}{\partial x_3} + \frac{q(\cos 2\psi + \varepsilon)}{r} = 0, \\
[\sin(\alpha - 2\psi)\sin\varphi + \sin\alpha]\,\frac{\partial p}{\partial x_1} + [\cos\alpha + \cos(\alpha - 2\psi)\sin\varphi]\,\frac{\partial p}{\partial x_3} \\
-2q\cos(\alpha - 2\psi)\,\frac{\partial \psi}{\partial x_1} + 2q\sin(\alpha - 2\psi)\,\frac{\partial \psi}{\partial x_3} - \frac{q\sin 2\psi}{r} = 0.
\end{gathered} \tag{9}$$

We resolve system (9) with respect to the derivatives $\partial p/\partial x_3$ and $\partial\psi/\partial x_3$ and obtain

$$\begin{gathered}
2q[\sin\varphi + \cos(2\alpha - 2\psi)]\,\frac{\partial \psi}{\partial x_3} - \cos^2\varphi\,\frac{\partial p}{\partial x_1} + 2q\sin(2\alpha - 2\psi)\,\frac{\partial \psi}{\partial x_1} \\
+ \frac{q}{r}\,[(\varepsilon + \sin\varphi)\cos\alpha + (1 + \varepsilon\sin\varphi)\cos(\alpha - 2\psi)] = 0, \\
[\sin\varphi + \cos(2\alpha - 2\psi)]\,\frac{\partial p}{\partial x_3} + \sin(2\alpha - 2\psi)\,\frac{\partial p}{\partial x_1} - 2q\,\frac{\partial \psi}{\partial x_1} - \frac{q}{r}\,[\varepsilon\sin(\alpha - 2\psi) + \sin\alpha] = 0.
\end{gathered} \tag{10}$$

The coefficients at $\partial p/\partial x_3$ and $\partial\psi/\partial x_3$ in these equations vanish if condition (7) is satisfied, and the equations themselves yield relations along the characteristics if $\partial p/\partial x_3$ and $\partial\psi/\partial x_3$ are bounded. Let us assume that the friction surface coincides with the envelope of the family of characteristics. Then, the characteristic relations should not be satisfied at the friction surface; to satisfy Eqs. (10), we have to assume that

$$\left|\frac{\partial p}{\partial x_3}\right| \to \infty, \qquad \left|\frac{\partial \psi}{\partial x_3}\right| \to \infty \tag{11}$$

as $\psi \to \alpha - \varphi/2 - \pi/4$ and $x_3 \to 0$. In the vicinity of the friction surface, we assume that

$$\psi = \alpha - \varphi/2 - \pi/4 + Ax_3^{\beta}. \tag{12}$$

For the second condition in (11) to be satisfied, the inequality $\beta < 1$ should be valid. At the same time, the boundedness of ψ requires that $\beta > 0$. Substituting (12) into the first equation in (10), we find that the first term of this equation has the order $O(x_3^{2\beta-1})$. As it was assumed that this term cannot be equal to zero (in this case, we would have a characteristic equation) and cannot tend to infinity [Eq. (10) does not contain other terms that could tend to infinity as $x_3 \to 0$], it follows that $\beta = 1/2$. Then, from Eq. (12), we obtain the dependence of the angle ψ in the vicinity of the friction surface on x_3 in the form

$$\psi = \alpha - \varphi/2 - \pi/4 + Ax_3^{1/2} + o(x_3^{1/2}). \tag{13}$$

Substituting Eq. (13) into the second equation of system (10) and using similar considerations, we obtain the expression

$$p = p_0 + Bx_3^{1/2} + o(x_3^{1/2}). \tag{14}$$

The values of A, B, and p_0 in Eqs. (13) and (14) can depend on the position of the point M on the friction surface.

The kinematic equations of the material model considered have the form [1]

$$\begin{gathered}\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0,\\ \Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big)\cos 2\psi - \Big(\frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z}\Big)\sin 2\psi\\ + \sin\varphi\Big(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} + 2u_r\,\frac{\partial \psi}{\partial r} + 2u_z\,\frac{\partial \psi}{\partial z} + 2\,\frac{\partial \psi}{\partial t}\Big) = 0.\end{gathered} \tag{15}$$

Here, u_r and u_z are the projections of the velocity vector onto the r and z axes, respectively, and t is the time. We introduce the absolute value of the velocity vector u and the angle γ between the r axis and the velocity vector, which is counted from the axis counterclockwise. Then, we have

$$u_r = u\cos\gamma, \qquad u_z = u\sin\gamma. \tag{16}$$

On the friction surface, for a chosen direction of the velocity vector (see Fig. 2), we have $\gamma = \alpha + \pi$. We assume that the behavior of γ in the vicinity of this surface is described by the function

$$\gamma = \alpha + \pi + Cx_3^{k}, \tag{17}$$

where k is a constant, and the value of C can depend on the position of the point M on the friction surface. Substituting (16) into (15), passing to differentiation with respect to x_1 and x_3 with the use of Eq. (8), resolving the resultant equations with respect to $\partial\gamma/\partial x_3$ and $\partial u/\partial x_3$, and using Eqs. (13) and (17), we obtain

$$\begin{gathered}Ax_3^{1/2}\,\frac{\partial u}{\partial x_3} + \frac{\partial u}{\partial x_1} + O(x_3^{k}) + O(x_3^{k-1/2}) + O(1) = 0,\\ ACukx_3^{k-1/2} + (Ax_3^{1/2} - Cx_3^{k})\,\frac{\partial u}{\partial x_1} + O(x_3^{2k-1/2}) + O(x_3^{k}) + O(x_3^{1/2}) = 0.\end{gathered} \tag{18}$$

In these equations, the terms of higher orders are omitted. Taking into account that u has the order $O(1)$ in the vicinity of the surface $x_3 = 0$ and comparing the exponent of the first term of the second equation in (18) with the exponents of other terms in this equation, we can conclude that $k = 1$. For such a value of k, it follows from the first equation of system (18) that the last term of this equation can be compensated only if

$$u = u_0 + Dx_3^{1/2} + o(x_3^{1/2}), \tag{19}$$

and Eq. (17) is written as

$$\gamma = \alpha + \pi + Cx_3. \tag{20}$$

Expressions (19) and (20) show that the singular character of the velocity field in the case considered is the same as in the classical theory of plasticity [7, 8] and in plane flows of materials obeying the double shear model [12]. In particular, the equivalent strain rate tends to infinity near the maximum friction surface in an inverse proportion to the square root from the distance to this surface. Such a behavior of the equivalent strain rate made it possible to introduce the concept of a strain-rate intensity coefficient [8, 12], which can be used to describe processes that occur in a thin layer in the vicinity of the friction surface [13, 14].

An analytica solution that describes the flow of a material obeying the double shear model through an infinite conical convergent channel and corresponds to regime A (see Fig. 1) was obtained in [1]. The law of the maximum friction was assumed to be valid on the channel walls. In the spherical coordinate system ρ, ϑ, θ determined by the relations $\rho^2 = r^2 + z^2$ and $\tan\vartheta = r/z$, the velocity field was obtained in the form $u_\vartheta = 0$ and

$$u_\rho = -V\rho^{-2}h(\vartheta), \tag{21}$$

where V is a constant and the function $h(\vartheta)$ is determined by the equation

$$\frac{dh}{d\vartheta} = -\frac{3h\sin 2\chi}{\cos 2\chi + \sin\varphi}. \tag{22}$$

In turn, χ is also a function of ϑ and is determined by the relation

$$\frac{d\chi}{d\vartheta} + 1 = \frac{n\cos^2\varphi\sin\vartheta - (1+\sin\varphi)\sin\varphi[\sin(2\chi+\vartheta) + \sin\vartheta]}{2\sin\vartheta\sin\varphi(\cos 2\chi + \sin\varphi)}, \tag{23}$$

where n is a constant. The physical meaning of the angle χ is the inclination of the first principal direction of the stress tensor to the ρ axis. On the friction surface, for $\vartheta = \vartheta_0$, we have

$$\chi = \varphi/2 + \pi/4. \tag{24}$$

Equation (23) should be solved under the boundary condition (24). It follows from Eqs. (21), (22), and (24) that the derivative of the radial velocity with respect to ϑ tends to infinity when approaching the friction surface. Hence, the velocity field is singular, and the equivalent strain rate near the friction surface, which is determined by the expression $\xi_{\rm eq} = \sqrt{2/3}(\xi_{ij}\xi_{ij})^{1/2}$ (ξ_{ij} are the components of the strain-rate tensor), is written in the form

$$\xi_{\rm eq} = \frac{1}{\sqrt{3}\rho}\Big|\frac{du_\rho}{d\vartheta}\Big| + \ldots. \tag{25}$$

Substituting Eq. (21) into (25) with allowance for (22) and decomposing the numerator and denominator of the resultant expression into a series with respect to χ in the vicinity of the point $\chi = \varphi/2 + \pi/4$, we obtain

$$\xi_{\rm eq} = \frac{\sqrt{3}Vh_0}{2\rho^3(\varphi/2 + \pi/4 - \chi)} + o\Big[\Big(\frac{\varphi}{2} + \frac{\pi}{4} - \chi\Big)^{-1}\Big], \tag{26}$$

where h_0 is the value of h on the friction surface. In the vicinity of the friction surface, the solution of Eq. (23) under the boundary condition (24) can be represented in the form

$$\Big(\chi - \frac{\varphi}{2} - \frac{\pi}{4}\Big)^2 = \frac{n\cos^2\varphi\sin\vartheta_0 - (1+\sin\varphi)\sin\varphi[\cos(\varphi+\vartheta_0) + \sin\vartheta_0]}{\sin\vartheta_0\sin 2\varphi}\,(\vartheta_0 - \vartheta). \tag{27}$$

Substituting (27) into (26), we can verify that

$$\xi_{\rm eq} = E(\rho)/[\rho(\vartheta_0 - \vartheta)]^{1/2} + o[(\vartheta_0 - \vartheta)^{-1/2}]. \tag{28}$$

Here E is the strain-rate intensity coefficient, which depends on ρ and on parameters of the process and material [8, 12]. Relation (28) shows that the behavior of the velocity field in the particular problem considered is in agreement with the global presentation of the velocity field (19).

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REFERENCES

1. A. J. M. Spencer, "Deformation of ideal granular materials," in: *Mechanics of Solids. The Rodney Hill 60th Anniversary Volume*, Pergamon Press, Oxford (1982), pp. 607–652.
2. A. Yu. Ishlinskii, "Plane motion of sand," *Ukr. Mat. Zh.*, **6**, No. 4, 430–441 (1954).
3. J. Ostrowska-Maciejewska and D. Harris, "Three-dimensional constitutive equations for rigid/perfectly plastic granular materials," *Math. Proc. Cambr. Philos. Soc.*, **108**, 153–169 (1990).
4. W. A. Spitzig, R. J. Sober, and O. Richmond, "The effect of hydrostatic pressure on the deformation behavior of maraging and HY-80 steels and its implications for plasticity theory," *Metallurg. Trans.*, **7A**, No. 11, 1703–1710 (1976).
5. A. S. Kao, H. A. Kuhn, W. A. Spitzig, and O. Richmond, "Influence of superimposed hydrostatic pressure on bending fracture and formability of a low carbon steel containing globular sulfides," *Trans. ASME, J. Eng. Mater. Technol.*, **112**, No. 1, 26–30 (1990).
6. V. V. Sokolovskii, "Equations of a plastic boundary-layer flow," *Prikl. Mat. Mekh.*, **20**, No. 3, 328–334 (1956).
7. S. E. Alexandrov, "Discontinuous velocity fields in an arbitrary deformation of an ideal rigid-plastic body," *Dokl. Ross. Akad. Nauk*, **324**, No. 4, 769–771 (1992).
8. S. Alexandrov and O. Richmond, "Singular plastic flow fields near surfaces of maximum friction stress," *Int. J. Non-Linear Mech.*, **36**, No. 1, 1–11 (2001).
9. C. S. Pemberton, "Flow of imponderable granular materials in wedge-shaped channels," *J. Mech. Phys. Solids*, **13**, 351–360 (1965).
10. E. A. Marshall, "The compression of a slab of ideal soil between rough plates," *Acta Mech.*, **3**, 82–92 (1967).
11. S. Alexandrov, "Comparison of double-shearing and coaxial models of pressure-dependent plastic flow at frictional boundaries," *Trans. ASME, J. Appl. Mech.*, **70**, No. 2, 212–219 (2003).
12. S. E. Alexandrov and E. A. Lyamina, "Singular solutions for a plane plastic flow of materials sensitive to the mean stress," *Dokl. Ross. Akad. Nauk*, **383**, No. 4, 492–495 (2002).
13. S. Alexandrov, "Interrelation between constitutive laws and fracture criteria in the vicinity of friction surfaces," in: *Physical Aspects of Fracture*, Kluwer, Dordrecht (2001), pp. 179–190.
14. S. E. Alexandrov, R. V. Goldstein, and E. A. Lyamina, "Development of the concept of the strain-rate intensity coefficient in the plasticity theory," *Dokl. Ross. Akad. Nauk*, **389**, No. 2, 180–183 (2003).